

On spectral decomposition of Smale-Vietoris axiom A diffeomorphisms ^{*†}

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Absrtact

We introduce Smale-Vietoris diffeomorphisms that include the classical DE-mappings with Smale solenoids. We describe the correspondence between basic sets of axiom A Smale-Vietoris diffeomorphisms and basic sets of nonsingular axiom A endomorphisms. For Smale-Vietoris diffeomorphisms of 3-manifolds, we prove the uniqueness of nontrivial solenoidal basic set. We construct a bifurcation between different types of solenoidal basic sets which can be considered as a destruction (or birth) of Smale solenoid.

Introduction

Stephen Smale, in his celebrated paper [20], introduced so-called DE-maps which arise from expanding maps (the abbreviation DE is formed by first letters of *Derived* from *Expanding* map). Let T be a closed manifold of dimension at least 1, and N an n -disk of dimension $n \geq 2$. Omitting details, one can say that a DE map is the skew map

$$f : T \times N \rightarrow T \times N, \quad (x; y) \mapsto (g_1(x); g_2(x, y)), \quad (1)$$

where $g_1 : T \rightarrow T$ is an expanding map of degree $d \geq 2$, and

$$g_2|_{\{x\} \times N} : \{x\} \times N \rightarrow \{g_1(x)\} \times N$$

an uniformly attracting map of n -disk $\{x\} \times N$ into n -disk $\{g_1(x)\} \times N$ for every $x \in T$. In addition, f must be a diffeomorphism onto its image $T \times N \rightarrow f(T \times N)$. In the particular case, when $T = S^1$ is a circle and $N = D^2$ is a 2-disk with the uniformly attracting g_2 , one gets a classical Smale solenoid $\cap_{l \geq 0} f^l(T \times D^2) = \mathfrak{S}$, see Fig. 1, that is a topological solenoid.

Recall that a topological solenoid was introduced by Vietoris [22] in 1927 (independently, a solenoid was introduced by van Danzig [7] in 1930, see review in [21]). Smale [20] proved that $\mathfrak{S}(f)$ is a hyperbolic expanding attractor. This construction was generalized by Williams [23, 24] who defined g_1 to be expansion mappings of branch manifolds (this allows to Williams to classify interior dynamics of expanding attractors) and by Block [2] who defined g_1 to be an axiom A endomorphism. The last paper concerns to the Ω -stability and the proving of

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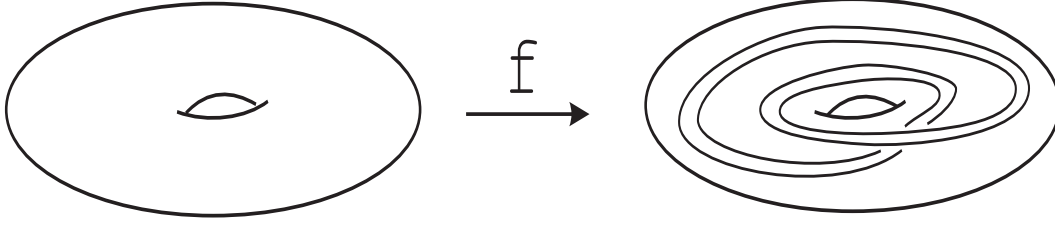


Fig. 1: DE-map by S. Smale

decomposition of non-wandering set into so-called basic sets (Spectral Decomposition Theorem for A-endomorphisms). Ideologically, our paper is a continuation of [2], where it was proved the following result (Theorem A). Let $f : M^n \rightarrow M^n$ be a Smale-Vietoris diffeomorphism of closed n -manifold M^n and $\mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}}$ (see the notations below). Then $f|_{\mathfrak{B}}$ satisfies axiom A on \mathfrak{B} if and only if g does on T .

Let us mention that in the frame of Smale-Williams construction the interesting examples of expanding attractors was obtained in [4, 8, 10, 13, 18]. Bothe [3] classified the purely Smale solenoids on 3-manifolds. He was first who also proved that a DE map $S^1 \times D^2 \rightarrow S^1 \times D^2$ can be extended to a diffeomorphism of some closed 3-manifold $M^3 \supset S^1 \times D^2$ (see also [5, 11, 12]). Ya. Zeldovich and others (see [6]) conjectured that Smale type mappings could be responsible for so-called fast dynamos. Therefore, it is natural to consider various generalizations of classical Smale mapping.

In a spirit of Smale construction of DE-maps, we here introduce diffeomorphisms called Smale-Vietoris that are derived from nonsingular endomorphisms. A non-wandering set of Smale-Vietoris diffeomorphisms belongs to an attractive invariant set of solenoidal type. In the classical case, the invariant set coincides with the non-wandering set consisting of a unique basic set. In general, the non-wandering set does not coincide with the invariant set, and divides into basic sets provided the nonsingular endomorphism is an A-endomorphism.

Let N be $(n - k)$ -dimensional compact Riemannian manifold with a non-empty boundary where $n - k \geq 1$. For a subset $N_1 \subset N$, we define the diameter $\text{diam } N_1 = \max_{a,b \in N_1} \{\rho_N(a, b)\}$ of N_1 where ρ_N is the metric on N . Denote by $\mathbb{T}^k = \underbrace{S^1 \times \cdots \times S^1}_k$ the k -dimensional torus,

$k \in \mathbb{N}$. A surjective mapping $g : \mathbb{T}^k \rightarrow \mathbb{T}^k$ is called a d -cover if g is a preserving orientation local homeomorphism of degree d . A good example is the preserving orientation linear expanding mapping $E_d : \mathbb{T}^k \rightarrow \mathbb{T}^k$ defined by an integer $k \times k$ matrix with the determinant equals d . Certainly, E_d is a d -cover.

A skew-mapping

$$F : \mathbb{T}^k \times N \rightarrow \mathbb{T}^k \times N, \quad (t, z) \mapsto ((g(t); \omega(t, z))) \quad (2)$$

is called a *Smale skew-mapping* is the following conditions hold:

- $F : \mathbb{T}^k \times N \rightarrow F(\mathbb{T}^k \times N)$ is a diffeomorphism on its image;
- $g : \mathbb{T}^k \rightarrow \mathbb{T}^k$ is a d -cover, $d \geq 2$;
- given any $t \in \mathbb{T}^k$, the restriction $w|_{\{t\} \times N} : \{t\} \times N \rightarrow \mathbb{T}^k \times N$ is the uniformly attracting embedding

$$\{t\} \times N \rightarrow \text{int}(\{g(t)\} \times N) \quad (3)$$

i.e., there are $0 < \lambda < 1$, $C > 0$ such that

$$\text{diam}(F^n(\{t\} \times N)) \leq C\lambda^n \text{diam}(\{t\} \times N), \quad \forall n \in \mathbb{N}. \quad (4)$$

When $g = E_d$, Smale skew-mapping is a DE mapping (1) introduced by Smale [20].

A diffeomorphism $f : M^n \rightarrow M^n$ is called a *Smale-Vietoris* diffeomorphism if there is the n -submanifold $\mathbb{T}^k \times N \subset M^n$ such that the restriction $f|_{\mathbb{T}^k \times N} \stackrel{\text{def}}{=} F$ is a Smale skew-mapping. The submanifold $\mathbb{T}^k \times N \subset M^n$ is called a *support of Smale skew-mapping*.

Put by definition,

$$\cap_{l \geq 0} F^l(\mathbb{T}^k \times N) \stackrel{\text{def}}{=} \mathfrak{S}(f).$$

One can easy to see that the set $\mathfrak{S}(f) = \mathfrak{S}$ is attractive, invariant and closed, so that the restriction

$$f|_{\mathfrak{S}} : \mathfrak{S} \rightarrow \mathfrak{S}$$

is a homeomorphism.

The following theorem shows that there is an intimate correspondens between basic sets of $f|_{\mathfrak{B}}$ and basic sets of the A-endomorphism g .

Theorem 1 *Let $f : M^n \rightarrow M^n$ be a Smale-Vietoris A-diffeomorphism of closed n -manifold M^n and $\mathbb{T}^k \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$, see (2). Let Ω be a basic set of $g : \mathbb{T}^k \rightarrow \mathbb{T}^k$ and $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^k \times N)$. Then $\mathfrak{S} \cap p_1^{-1}(\Omega)$ contains a unique basic set $\Omega_{\mathfrak{S}}$ of f . Here, $p_1 : \mathbb{T}^k \times N \rightarrow \mathbb{T}^k$ is the natural projection on the first factor. Moreover,*

1. *If Ω is a trivial basic set (isolated periodic orbit) of g then $\Omega_{\mathfrak{S}}$ is also trivial basic set.*
2. *If Ω is a nontrivial basic set of g then $\Omega_{\mathfrak{S}}$ is also nontrivial basic set.*
3. *If Ω is a backward g -invariant basic set of g , $\Omega = g^{-1}(\Omega)$, (hence, Ω is nontrivial) then $\Omega_{\mathfrak{S}} = \mathfrak{S} \cap p_1^{-1}(\Omega)$.*

For $k = 1$, when $\mathbb{T}^1 = S^1$ is a circle, the following result says that $NW(F)$ contains a unique nontrivial basic set that is either Smale (one-dimensional) solenoid or a nontrivial zero-dimensional basic set.

Theorem 2 *Let $f : M^n \rightarrow M^n$ be a Smale-Vietoris A-diffeomorphism of closed n -manifold M^n and $\mathbb{T}^1 \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$. Then the non-wandering set $NW(F)$ of F belongs to $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^1 \times N)$, and $NW(F)$ contains a unique nontrivial basic set $\Lambda(f)$ that is either*

- *a one-dimensional expanding attractor, and $\Lambda(f) = \mathfrak{S}$, or*
- *a zero-dimensional basic set, and $NW(F)$ consists of $\Lambda(f)$ and finitely many (nonzero) isolated attracting periodic points plus finitely many (possibly, zero) saddle type isolated periodic points of codimension one stable Morse index.*

The both possibilities hold.

It is natural to consider bifurcations from one type of dynamics to another which can be thought of as a destruction (or, a birth) of Smale solenoid. For simplicity, we represent two such bifurcations for $n = 3$ and $M^3 = S^3$ a 3-sphere. Recall that a diffeomorphism $f : M \rightarrow M$ is Ω -stable if there is a neighborhood $U(f)$ of f in the space of C^1 diffeomorphisms $\text{Diff}^1(M)$ such that $f|_{NW(f)}$ conjugate to every $g|_{NW(g)}$ provided $g \in U(f)$.

Theorem 3 *There is the family of Ω -stable Smale-Vietoris diffeomorphisms $f_\mu : S^3 \rightarrow S^3$, $0 \leq \mu \leq 1$, continuously depending on the parameter μ such that the non-wandering set $NW(f_\mu)$ of f_μ is the following:*

- $NW(f_0)$ consists of a one-dimensional expanding attractor (Smale solenoid attractor) and one-dimensional contracting repeller (Smale solenoid repeller);
- for $\mu > 0$, $NW(f_\mu)$ consists of two nontrivial zero-dimensional basic sets and finitely many isolated periodic orbits.

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1 Definitions

A mapping $F : M \times N \rightarrow M \times N$ of the type $F(x; y) = (g(x); h(x, y))$ is called a *skew-mapping*. One says also a *skew product transformation* over g or simply, a *skew product*. Denote by $\text{End}(M)$ the space of C^1 endomorphisms $M \rightarrow M$ i.e., the C^1 maps of M onto itself. An endomorphism g is *nonsingular* if the Jacobian $|Dg| \neq 0$. This means that g is a local diffeomorphism. In particular, g is a d -cover. In this paper, we consider nonsingular $g \in \text{End}(M)$, $Dg \neq 0$, that are not a diffeomorphism.

Fix $g \in \text{End}(M)$. A point $x \in M$ is said to be *non-wandering* if given any neighborhood $U(x) = U$ of x , there is $m \in \mathbb{N}$ such that $g^m(U) \cap U \neq \emptyset$. Denote by $NW(g)$ the set of non-wandering points. Clearly, $NW(g)$ is a closed set and $g(NW(g)) \subset NW(g)$ i.e., $NW(g)$ is a forward g -invariant set. The set $\{x_i\}_{-\infty}^{\infty}$ denoted by $O(x_0)$ is called a g -orbit of x_0 if $g(x_i) = x_{i+1}$ for every integer i . A subset $\{x_j, x_{j+1}, \dots, x_{j+r}\} \subset O(x_0)$ consisting of a finitely many points of $O(x_0)$ is called a *compact part* of $O(x_0)$. A g -orbit $\{x_i\}_{-\infty}^{\infty}$ is *periodic* if there is an integer $p \geq 0$ such that $g^p(x_i) = x_{i+p}$ for each $i \in \mathbb{Z}$. Certainly, $NW(g)$ contains all periodic g -orbits.

The orbit $O(x_0)$ is said to be *hyperbolic* if there is a continuous splitting of the tangent bundle

$$\mathbb{T}_{O(x_0)}M = \bigcup_{i=-\infty}^{\infty} \mathbb{T}_{x_i}M = \mathbb{E}^s \oplus \mathbb{E}^u = \bigcup_{i=-\infty}^{\infty} \mathbb{E}_{x_i}^s \oplus \mathbb{E}_{x_i}^u$$

which is preserved by the derivative Dg such that

$$\|Dg^m(v)\| \leq c\mu^m\|v\|, \|Dg^m(w)\| \geq c^{-1}\mu^{-m}\|w\| \quad \text{for } v \in \mathbb{E}^s, w \in \mathbb{E}^u, \forall m \in \mathbb{N}$$

for some constants $c > 0$, $0 < \mu < 1$ and a Riemannian metric on $\mathbb{T}M$. Note that $\mathbb{E}^u(x_0)$ depends on the negative semi-orbit $\{x_i\}_{i=-\infty}^0$. It may happen that $\mathbb{E}^u(x_0) \neq \mathbb{E}^u(y_0)$ though $x_0 = y_0$ but $O(x_0) \neq O(y_0)$. Such a phenomenon is impossible for $\mathbb{E}^s(x_0)$, it depends only on x_0 [16].

We say that a nonsingular $g \in \text{End}(M)$ satisfies axiom A, in short, f is an A-endomorphism if

- the periodic g -orbits are dense in $NW(g)$ (it follows that $g(NW(g)) = NW(g)$);
- all g -orbits of $NW(g)$ are hyperbolic, and the corresponding splitting of the tangent bundle $\mathbb{T}_{NW(g)}$ depends continuously on the compact parts of the g -orbits.

Recall that Smale's Spectral Decomposition Theorem says that for Axiom A diffeomorphisms the non-wandering set partitions into nonempty closed invariant sets each of which is transitive. Similar theorem for A-endomorphisms was probed in [2] (Theorem C), [16] (Theorem 3.11 and Proposition 3.13). Thus, if g is a nonsingular A-endomorphism then the non-wandering set $NW(g)$ is the disjoint union $\Omega_1 \cup \dots \cup \Omega_k$ such that each Ω_i is closed and invariant, $g(\Omega_i) = \Omega_i$, and Ω_i contains a point whose g -orbit is dense in Ω_i . The Ω_i are called *basic sets*.

Following Williams [23, 24], we introduce an inverse limit for $g : T \rightarrow T$ as follows. Put by definition, $\prod_g = \{(t_0, t_1, \dots, t_i, \dots) \in T^{\mathbb{N}} : g(t_{i+1}) = t_i, i \geq 0\}$. This set is endowed by the product topology of countable factors. This topology has a basis generating by (ε, r) -neighborhoods

$$U = \{ \{x_i\}_0^\infty \in \prod_g : x_i \in U_\varepsilon(t_i), 0 \leq i \leq r \text{ for some } \varepsilon > 0, r \in \mathbb{N} \}, \quad (5)$$

where $\{t_0, t_1, \dots, t_i, \dots\} \in \prod_g$. Define the shift map

$$\hat{g} : \prod_g \rightarrow \prod_g, \quad \hat{g}(t_0, t_1, \dots, t_i, \dots) = (g(t_0), t_0, t_1, \dots, t_i, \dots), \quad (t_0, t_1, \dots, t_i, \dots) \in \prod_g.$$

This map $\hat{g} : \prod_g \rightarrow \prod_g$ is called the *inverse limit* of g is a homeomorphism [17, 24].

2 Proofs of main results

We denote by $p_1 : \mathbb{T}^k \times N \rightarrow \mathbb{T}^k$, $p_2 : \mathbb{T}^k \times N \rightarrow N$ the natural projections $p_1(t, z) = t$ and $p_2(t, z) = z$. A fiber $\{t\} \times N \stackrel{\text{def}}{=} N_t$ of the trivial fiber bundle p_1 is called a t -leaf. It follows from (2) that $F = f|_{\mathfrak{B}}$ takes a t -leaf into $g(t)$ -leaf.

Let $t \in \mathbb{T}^k$ and $\varepsilon > 0$. We denote by $U_\varepsilon(t)$ the ε -neighborhood of the point t i.e., $U_\varepsilon(t) = \{x \in \mathbb{T}^k : \varrho(x, t) < \varepsilon\}$ where ϱ is a metric on \mathbb{T}^k .

The following technical lemma describes the symbolic model of the restriction $f|_{\mathfrak{S}}$. This lemma is a generalization of the similar classical result by Williams [23, 24].

Lemma 1 *Let $f : M^n \rightarrow M^n$ be a Smale-Vietoris diffeomorphism of closed n -manifold M^n and $\mathbb{T}^k \times N = \mathfrak{B} \subset M^n$ the support of Smale skew-mapping $f|_{\mathfrak{B}} = F$. Then the restriction $f|_{\mathfrak{S}}$ is conjugate to the inverse limit of the mapping $g : \mathbb{T}^k \rightarrow \mathbb{T}^k$, where $\mathfrak{S} = \cap_{l \geq 0} F^l(\mathbb{T}^k \times N)$.*

Proof. Recall that given any point $t_0 \in \mathbb{T}^k$, $g^{-1}(t_0)$ consists of d points, one says $t_0^1, t_0^2, \dots, t_0^d \in \mathbb{T}^k$. Since F is a diffeomorphism on its image, the sets $F(N_{t_0^1}), \dots, F(N_{t_0^d})$ are pairwise disjoint,

$$F(N_{t_0^i}) \cap F(N_{t_0^j}) = \emptyset, \quad i \neq j, \quad 1 \leq i, j \leq d, \quad (6)$$

Now, for the sake of simplicity, we divide the proof into several steps. The end of the proof of each step will be denoted by \diamond .

Step 1 Given any point $p \in \mathfrak{S}$, there is a unique sequence of points $\{t_i\}_{i=0}^\infty$, $t_i \in \mathbb{T}^k$, and the corresponding sequence of the leaves $\{N_{t_i}\}_{i=0}^\infty$ such that

- $p \in \cdots \subset F^i(N_{t_i}) \subset F^{i-1}(N_{t_{i-1}}) \cdots \subset F(N_{t_1}) \subset N_{t_0}$, $p = \bigcap_{i \geq 0} F^i(N_{t_i})$;
- $t_i = g(t_{i+1})$, $i \geq 0$.

Proof of Step 1. Put $t_0 = p_1(p) \in \mathbb{T}^k$. Let $g^{-1}(t_0) = \{t_0^1, t_0^2, \dots, t_0^d\}$. By (6), there is a unique t_0^j such that $p \in F(N_{t_0^j})$. Put by definition $t_0^j = t_1$. Note that $F(N_{t_1}) \subset N_{t_0}$. Similarly, $g^{-1}(t_1)$ consists of d points $t_1^1, t_1^2, \dots, t_1^d$. By (6), the sets $F(N_{t_1^1}), \dots, F(N_{t_1^d})$ are pairwise disjoint. Since $p \in F^2(\mathbb{T}^k \times N)$, there is a unique t_1^i such that $p \in F^2(N_{t_1^i})$. Put by definition $t_1^i = t_2$. Note that $p \in F^2(N_{t_2}) \subset F(N_{t_1}) \subset N_{t_0}$. Continuing by this way, one gets the sequences $\{t_i\}_{i=0}^\infty$, $\{N_{t_i}\}_{i=0}^\infty$ desired. It follows from (4) that $\text{diam } F^i(N_{t_i}) = \text{diam } (F^i(\{t_i\} \times N)) \rightarrow 0$ as $i \rightarrow \infty$. Hence, $p = \bigcap_{i \geq 0} F^i(N_{t_i})$. \diamond

Let $\hat{g} : \prod_g \rightarrow \prod_g$ be the inverse limit of $g : \mathbb{T}^k \rightarrow \mathbb{T}^k$ where $\prod_g = \{(t_0, t_1, \dots, t_i, \dots) \in \mathbb{T}^\mathbb{N} : g(t_{i+1}) = t_i, i \geq 0\}$. For a point $p \in \mathfrak{S}$, denote by $P(t_0, t_1, \dots, t_i, \dots)$, $t_i \in \mathbb{T}^k$, the sequence due to Step 1. Define the mapping

$$\theta : \mathfrak{S} \rightarrow \prod_g, \quad p \mapsto P(t_0, t_1, \dots, t_i, \dots), \quad p \in \mathfrak{S}.$$

Step 2 The mapping θ is a homeomorphism.

Proof of Step 2. It follows from (4) that θ is injective. Since the intersection of nested sequence of closed subsets is non empty, θ is surjective. One remains to prove that θ and θ^{-1} are continuous. Take a neighborhood U of $\theta(p)$, $p \in \mathfrak{S}$. We can assume that U is an (ε, r) -neighborhood (5), where $\theta(p) = \{t_0, t_1, \dots, t_i, \dots\} \in \prod_g$. Moreover, one can assume that $g^{-1}(U_\varepsilon(t_i))$ consists of d pairwise disjoint domains for every $0 \leq i \leq r$. Recall that $t_i = g(t_{i+1})$, $i \geq 0$. Therefore, $t_{r-j} = g^j(t_r)$ for all $1 \leq j \leq r$. Similarly, $x_{r-j} = g^j(x_r)$, $1 \leq j \leq r$. Since g is continuous, there exists $0 < \delta \leq \varepsilon$ such that the inclusion $x_r \in U_\delta(t_r)$ implies $x_i \in U_\varepsilon(t_i)$ for all $i = 0, \dots, r$. The restriction $F|_{\mathfrak{S}} : \mathfrak{S} \rightarrow \mathfrak{S}$ is a diffeomorphism. Therefore, there is a (relative) neighborhood $U(p)$ of p in \mathfrak{S} such that $p_1(F^{-i}(U(p))) \subset U_\delta(t_i)$ for all $0 \leq i \leq r$. Taking in mind that $g^{-1}(U_\varepsilon(t_i))$ consists of d pairwise disjoint domains, $0 \leq i \leq r$, we see that $\theta(U(p)) \subset U$. Thus, θ is continuous. Since \prod_g is compact, θ^{-1} is also continuous. \diamond

Step 3 One holds $\theta \circ F|_{\mathfrak{S}} = \hat{g} \circ \theta|_{\mathfrak{S}}$.

Proof of Step 3. Take $p \in \mathfrak{S}$ and $\theta(p) = \{t_0, t_1, \dots, t_i, \dots\}$ where $t_i = g(t_{i+1})$, $i \geq 0$. By definition of $\hat{g} : \prod_g \rightarrow \prod_g$, one holds $\hat{g} \circ \theta(p) = \hat{g}(\theta(p)) = \hat{g}(\{t_0, t_1, \dots, t_i, \dots\}) = \{g(t_0), t_0, t_1, \dots, t_i, \dots\}$. It follows from (3) that $F(p) \in F(\{t_0\} \times N) \subset N_{g(t_0)}$. Hence, by Step 1, the sequence of points $\{g(t_0), t_0, t_1, \dots, t_i, \dots\}$ corresponds to $\theta(F(p))$, since

$$\begin{aligned} F(p) &= F\left(\bigcap_{i \geq 0} F^i(\{t_i\} \times N)\right) = \bigcap_{i \geq 0} F^{i+1}(\{t_i\} \times N) = \bigcap_{i \geq 0} F^{i+1}(\{t_i\} \times N) \cap N_{g(t_0)} = \\ &= N_{g(t_0)} \cap F(N_{t_0}) \cap F^2(N_{t_1}) \cap \dots \cap F^{i+1}(N_{t_i}) \cap \dots \diamond \end{aligned}$$

It follows from Steps 2, 3 that the mapping θ is a conjugacy between $F|_{\mathfrak{S}}$ and \hat{g} . Lemma 1 is proved. \square

To prove Theorem 1 we need some previous results.

Lemma 2 Let $\bar{t} = \{t_0, t_1, \dots, t_i, \dots\} \in \prod_g$, $g(t_{i+1}) = t_i$, $i \geq 0$. Suppose that $t_i \in NW(g)$ for all $i \geq 0$. Then $\bar{t} \in NW(\hat{g})$ and $\theta^{-1}(\bar{t}) \in NW(F)$.

Proof. Since $\bar{t} = \{t_0, t_1, \dots, t_i, \dots\} = \{g^r(t_r), g^{r-1}(t_r), \dots, t_r, \dots\}$, we can take the (ε, r) -neighborhood V as follows

$$V = \{ \{g^r(x_r), g^{r-1}(x_r), \dots, x_r, \dots\} : g^i(x_r) \in U_\varepsilon(g^i(t_r)), 0 \leq i \leq r \}.$$

Since g, g^2, \dots, g^r are uniformly continuous, there is $0 < \delta \leq \varepsilon$ such that $x \in U_\delta(y)$ implies $g^i(x) \in U_\varepsilon(g^i(y))$ for all $0 \leq i \leq r$. By condition, $t_r \in NW(g)$. Hence, there exists $n_0 \in \mathbb{N}$ such that $g^{n_0}(V_\delta(t_r)) \cap V_\delta(t_r) \neq \emptyset$. It follows that there is a point $x_0 \in V_\delta(t_r)$ such that $g^{n_0}(x_0) \in V_\delta(t_r)$.

Take $\bar{x}_0 = \{g^r(x_0), g^{r-1}(x_0), \dots, x_0, \dots\} \in \prod_g$. Since $x_0 \in V_\delta(t_r)$, $g^i(x_0) \in U_\varepsilon(g^i(t_r))$ for all $0 \leq i \leq r$. Therefore, $\bar{x}_0 \in V$. Since $g^{n_0}(x_0) \in V_\delta(t_r)$, $g^{n_0+i}(x_0) \in U_\varepsilon(g^i(t_r))$ for all $0 \leq i \leq r$. Therefore,

$$\hat{g}^{n_0}(\bar{x}_0) = \{g^{n_0+r}(x_0), g^{n_0+r-1}(x_0), \dots, g^{n_0}(x_0), \dots\} \in V.$$

As a consequence, $\hat{g}^{n_0}(V) \cap V \neq \emptyset$ and $\bar{t} \in NW(\bar{g})$. A conjugacy map takes a non-wandering set onto non-wandering set. By Lemma 1, $\theta^{-1}(\bar{t}) \in NW(F)$. \square

Corollary 1 The following qualities hold $p_1[NW(f_{\mathfrak{B}})] = p_1[NW(F)] = NW(g)$.

Proof. Since the projection p_1 is continuous, $p_1[NW(F)] \subset NW(g)$. Take a point $t_0 \in NW(g)$. Since g is an A-endomorphism, $g[NW(g)] = NW(g)$ [2, 16]. Therefore, there is a sequence $t_i \in NW(g)$ such that $g(t_{i+1}) = t_i$ for every $i \geq 0$. It follows from Lemma 2 that $\bar{t} = \{t_0, t_1, \dots, t_i, \dots\} \in NW(\hat{g})$ and $\theta^{-1}(\bar{t}) \in NW(F)$. By definition of the mapping θ , $\theta^{-1}(\bar{t}) \in p_1^{-1}(t_0)$. Hence, $NW(g) \subset p_1[NW(F)]$. \square

Lemma 3 Let $(t_0, z_0) \in \mathfrak{S}$ be a non-wandering point of f , and $\theta(t_0, z_0) = \{t_i\}_{i \geq 0}$. Then $t_i \in NW(g)$ for all $i \geq 0$.

Proof. According to Corollary 1, $p_1[NW(f_{\mathfrak{B}})] = p_1[NW(F)] = NW(g)$. Therefore, $t_0 \in NW(g)$. Since $F_{\mathfrak{S}} : \mathfrak{S} \rightarrow \mathfrak{S}$ is a diffeomorphism, $F^{-1}(NW(F)) = NW(F)$ and $F^{-1}(t_0, z_0) = (t_1, z_1) \in NW(F) = NW(f_{\mathfrak{B}})$. Hence, $t_1 \in NW(g)$ by Step 1. Continuing this way, one gets that $t_i \in NW(g)$ for all $i \geq 0$. \square

Corollary 2 Let $(t_0, z_0) \in \mathfrak{S}$ be a non-wandering point of f , and $\theta(t_0, z_0) = \{t_i\}_{i \geq 0}$. Suppose that t_0 belongs to a basic set Ω of g . Then $t_i \in \Omega$ for all $i \geq 0$.

Proof. By Lemma 3, $t_i \in NW(g)$ for all $i \geq 0$. Since Ω is forward g -invariant, $t_i \in \Omega$ for all $i \geq 0$. \square

Lemma 4 Let Ω be a nontrivial basic set of g , and $t_0 \in \Omega$. Suppose that two points $(t_0, z_1), (t_0, z_2) \in \mathfrak{S}$ are non-wandering under f . Then the both (t_0, z_1) and (t_0, z_2) belong to the same basic set of f .

Proof. Denote by Ω_j the basic set of F containing the point (t_0, z_j) , $j = 1, 2$. Clearly, $\Omega_j \subset \mathfrak{S}$. We have to prove that $\Omega_1 = \Omega_2$. It is sufficient to show that there is a non-wandering point $q \in NW(F)$ such that each point (t_0, z_1) and (t_0, z_2) belongs to the ω -limit set of q .

Let $\bar{t}_j = \theta(t_0, z_j) = \{t_0, t_1^{(j)}, \dots, t_i^{(j)}, \dots\}$, $j = 1, 2$. By Corollary 2, $t_i^{(j)} \in \Omega$ for all $i \geq 0$, $j = 1, 2$. Since the basic set Ω is transitive, there is a point $x_0 \in \Omega$ such that its positive semi-orbit $O_g^+(x_0)$ is dense in Ω , $\text{clos}(O_g^+(x_0)) = \Omega$.

It follows from Corollary 1 that there is a point $\bar{x}_0 = \{x_0, x_1, \dots, x_i, \dots\} \in \prod_g$ such that $x_i \in \Omega$ for all $i \geq 0$. Take arbitrary (ε, r) -neighborhood $U(\bar{t}_1)$ of \bar{t}_1 . Since g, g^2, \dots, g^r are uniformly continuous, there exists $\delta > 0$ such that the inequality $x \in U_\delta(y)$ implies $g^i(x) \in U_\varepsilon(y)$ for all $0 \leq i \leq r$. Because of the semi-orbit $O_g^+(x_0)$ is dense in Ω , there is $n_0 \in \mathbb{N}$ such that $g^{n_0}(x_0) \in U_\delta(t^{(1)})$. Hence, $\hat{g}^{n_0}(\bar{x}_0) \in U(\bar{t}_1)$. Therefore, $\bar{t}_1 = \theta(t_0, z_1)$ belongs to the ω -limit set of \bar{x}_0 . Similarly, one can prove that $\bar{t}_2 = \theta(t_0, z_2)$ belongs to the ω -limit set of \bar{x}_0 as well. Since θ is a conjugacy mapping, the points $(t_0, z_1) = \theta^{-1}(\bar{t}_1)$ and $(t_0, z_2) = \theta^{-1}(\bar{t}_2)$ belongs to the ω -limit set of the point $q = \theta^{-1}(\bar{x}_0) \in NW(F)$. \square

Proof of Theorem 1. We know that $p_1[NW(F)] = NW(g)$. Hence, $\mathfrak{S} \cap p_1^{-1}(\Omega)$ contains basic sets of f . Suppose that Ω is trivial i.e., Ω is an isolated periodic orbit

$$\Omega = Orb_g(q) = \{q, g(q), \dots, g^{p-1}(q), g^p(q) = q\}, \quad \text{where } q \in \mathbb{T}^k \text{ and } p \in \mathbb{N} \text{ is a period of } q.$$

By definition of Smale skew-mapping, the restriction of $F = f|_{\mathfrak{B}}$ on the second factor N is the uniformly attracting embedding. Therefore,

$$N_q \supset f^p(N_q) \supset \dots \supset f^{mp}(N_q) \supset \dots \text{ and the intersection } \bigcap_{m \geq 0} f^{mp}(N_q) \text{ is a unique point, say } Q.$$

Similarly, $\bigcap_{m \geq 0} f^{mp}(N_{g^i(q)})$ is a unique point $f^i(Q)$ for every $0 \leq i \leq p-1$. It follows from (2), that $\{Q, f(Q), \dots, f^{p-1}(Q), f^p(Q) = Q\}$ is an isolated periodic orbit $Orb_f(Q)$ such that $NW(F) \cap p_1^{-1}(\Omega) = Orb_f(Q)$. Therefore, $Orb_f(Q) = \Omega_{\mathfrak{S}}$ is a unique basic set of F that belongs to $\mathfrak{S} \cap p_1^{-1}(\Omega)$.

Let Ω be a nontrivial basic set. It follows from Lemma 4 that all basic set of F that is contained in $\mathfrak{S} \cap p_1^{-1}(\Omega)$ are coincide. Hence, $\Omega_{\mathfrak{S}}$ is a unique nontrivial basic set of f contained in $\mathfrak{S} \cap p_1^{-1}(\Omega)$.

Now let Ω be a backward g -invariant basic set of g . Note that the equality $\Omega = g^{-1}(\Omega)$ implies that Ω cannot be a trivial basic set, since g is a d -cover, $d \geq 2$. It follows from Lemma 2 that every point of $\mathfrak{S} \cap p_1^{-1}(\Omega)$ is a non-wandering point of f . By Lemma 4, $\mathfrak{S} \cap p_1^{-1}(\Omega)$ is a unique basic set. Theorem 1 is proved. \square

Example. Let us consider three endomorphism $g_i : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $i = 1, 2, 3$, that are 2-covers. g_1 is defined by the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Clearly, g_1 is an expanding A-endomorphism, and \mathbb{T}^2 is a unique basic set of g_1 . The corresponding diffeomorphism f has a unique basic set, say Ω_1 , that is locally homeomorphic to the product of \mathbb{R}^2 and Cantor set. Thus, Ω_1 is 2-dimensional.

Now, let us consider the case when $\mathbb{T}^1 = S^1$ is a circle, and d -cover $g : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is a nonsingular endomorphism of S^1 . The crucial step of the proof of Theorem 2 is the following result.

Lemma 5 *Let $g : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ be a nonsingular A-endomorphism, and $NW(g)$ a non-wandering set of g . Then $NW(g)$ is either \mathbb{T}^1 or $NW(g)$ is the union of the Cantor type set Σ and finitely many (nonzero) isolated attracting periodic orbits plus finitely many (possibly, zero) repelling isolated periodic orbits. Moreover, in the last case, Σ is backward g -invariant.*

Proof. Suppose that $NW(g) \neq \mathbb{T}^1$. By [19], g is semi-conjugate to the expanding linear mapping E_d , $E_d(t) = dt \bmod 1$, i.e., there is a continuous map $h : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ such that $g \circ h = h \circ E_d$. Moreover, h is monotone [14]. As a consequence, given any point $t \in \mathbb{T}^1$, $h^{-1}(t)$ is either a point or a closed segment. Since $NW(g) \neq \mathbb{T}^1$, h is not a homeomorphism. Hence, there are points $t \in \mathbb{T}^1$ for which $h^{-1}(t)$ is a (nontrivial) closed segment. Denote the set of such points by χ . The set χ is countable and invariant under E_d , $E_d(\chi) = E_d^{-1}(\chi) = \chi$ [1, 14]. Therefore, $h^{-1}(\chi)$ is also invariant under g . As a consequence, $\Sigma = \mathbb{T}^1 \setminus \text{clos}(h^{-1}(\chi))$ is the Cantor set consisting on non-wandering points of g . Moreover, Σ is invariant under g (in particular, backward g -invariant). It follows from [15] that the part of $NW(g)$ that different from Σ consists of finitely many (nonzero) isolated attracting periodic orbits and finitely many (possibly, zero) repelling isolated periodic orbits. \square

Now, Theorem 2 except the realization part immediately follows from Theorem 1 and Lemma 5. It remains to construct a Smale-Vietoris A-diffeomorphism the non-wandering set of whose consists of a nontrivial zero-dimensional basic set and a finitely many (nonzero) isolated periodic orbits. It follows from [3, 5] for $n = 3$ and [2, 23] for $n \geq 4$ that it is sufficient to construct Smale skew-mapping $F : S^1 \times D^{n-1} \rightarrow S^1 \times D^{n-1}$ with the non-wandering set desired because of Smale skew-mapping can be extended to a diffeomorphism of some closed n -mannifold. Moreover, according to Robinson-Williams [18] construction of classical Smale solenoid, we can suppose $n = 3$.

Let $g : S^1 \rightarrow S^1$ be a C^∞ nonsingular A-endomorphism that is a d -cover ($d \geq 2$) with the non-wandering set $NW(g)$ consisting of a unique attracting fixed point x_0 and a Cantor set Ω . Moreover, one can assume that $Dg|_\Omega = 2d - 1$, $Dg(x_0) = \lambda < 1$ where λ will be specified below. Such endomorphism was constructed by Shub [19]. Hirsch [9] has noticed that such endomorphism can be smoothed to be analytical. Now, the circle S^1 is endowed with the parameter inducing by the natural projection $[0; 1] \rightarrow [0; 1]/(0 \sim 1) = S^1$. We can assume that the restriction $g|_{[0; \frac{1}{2}]}$ is a diffeomorphism $[0; \frac{1}{2}] \rightarrow [0; \frac{1}{2}]$ with the attracting fixed point $x_0 = \frac{1}{4}$ and two repelling fixed points $0, \frac{1}{2}$. Without loss of generality, one can also assume that $g|_{[\frac{1}{2}; 1]}(x) = (2d - 1)x \bmod 1$. By construction, $\cup_{n \geq 0} g_d^{-n}(0; \frac{1}{2})$ is the stable manifold $W^s(x_0)$ of x_0 , and $\Omega = S^1 \setminus W^s(x_0)$ is Cantor set belonging to $NW(g)$. Clearly, given any $y \in S^1$, $\min_{t_k, t_j} \{|t_k - t_j| = \frac{1}{2d-1}$ where $t_k \neq t_j$ and $g(t_k) = g(t_j) = y$. We take $0 < \lambda < \frac{1}{4} \sin \frac{\pi}{2d-1}$. After this specification, we denote g by g_d . Put by definition

$$F(t, z) = \left(g_d(t), \lambda z + \frac{1}{2} \exp 2\pi i t \right), \quad F : \mathcal{B} = S^1 \times D^2 \rightarrow \mathcal{B}, \quad (7)$$

where $D^2 \subset \mathbb{R}^2$ is the unit disk, and $z = x + iy$, and \mathcal{B} is a support of Smale skew-mapping. Since $\lambda < \frac{1}{4}$, $F(\mathcal{B}) \subset \text{int } \mathcal{B}$. The Jacobian of F equals

$$DF(t, z) = \begin{pmatrix} Dg_d(t) & 0 \\ \pi i \exp 2\pi i t & \lambda Id_2 \end{pmatrix}, \quad (8)$$

where Id_2 is the identity matrix on \mathbb{C} or \mathbb{R}^2 . Since $Dg_d > 0$ and $\lambda > 0$, F is a local diffeomorphism. It follows from $\lambda < \frac{1}{4} \sin \frac{\pi}{2d-1}$ that F is a (global) diffeomorphism on its image.

Since g_d is an A-endomorphism, the periodic points of g_d are dense in $NW(g_d)$. By Lemma 2, the periodic points of F are dense in $NW(F)$. Thus, it remains to prove the $NW(F)$ has a hyperbolic structure. We follow [17], Proposition 8.7.5. Clearly, the tangent bundle $T(\mathcal{B}) = T(S^1 \times D^2)$ is the sum $T(\mathcal{B}) = T(S^1) \oplus T(D^2)$, and the fiber $T_{(t,z)}(\mathcal{B})$ at each point $(t, z) \in \mathcal{B}$ is the

sum of one-dimensional and two-dimensional tangent spaces $T_t(S^1) = \mathbb{E}^1 \cong \mathbb{R}$, $T_z(T^2) = \mathbb{E}^2 \cong \mathbb{R}^2$ respectively. It follows from (8) that \mathbb{E}^2 is invariant under DF :

$$DF_p \begin{pmatrix} \vec{0} \\ \vec{v}_{23} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \lambda \vec{v}_{23} \end{pmatrix}, \quad \vec{v}_{23} \in \mathbb{E}^2.$$

Moreover, since $|\lambda| < 1$, \mathbb{E}^2 is the stable bundle, $E^s = \mathbb{E}^2$.

Take $q = (t, z) \in NW(F)$. Then $p_1(q) = t \in NW(g_d)$. If $t = x_0$, then q is a hyperbolic (attractive) fixed point of F . For $t \in \Omega$, we consider the cones

$$C_q^u = \left\{ \begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix} : \vec{v}_1 \in T_t(S^1), \vec{v}_{23} \in \mathbb{E}_z^2, \quad |\vec{v}_1| \geq \frac{2d-1}{4} |\vec{v}_{23}| \right\} \subset T(\mathcal{B}) = \mathbb{E}^1 \oplus \mathbb{E}^2.$$

For $\begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix} \in C_q^u$, it follows from (8) that

$$DF \begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix} = \begin{pmatrix} \vec{v}_1' \\ \vec{v}_{23}' \end{pmatrix} = \begin{pmatrix} (2d-1)\vec{v}_1 \\ \pi i \vec{v}_1 \exp 2\pi i t + \lambda \vec{v}_{23} \end{pmatrix}.$$

Hence, $|\vec{v}_{23}| \leq |\pi i \exp 2\pi i t \vec{v}_1| + \lambda |\vec{v}_{23}| = \pi |\vec{v}_1| + \lambda |\vec{v}_{23}|$. Taking in mind $\lambda \leq \frac{1}{4}$, one gets

$$\begin{aligned} |\vec{v}_1'| &= (2d-1)|\vec{v}_1| = \frac{2d-1}{4}(4|\vec{v}_1|) \geq \frac{2d-1}{4} \left(\pi |\vec{v}_1| + \frac{1}{2} |\vec{v}_1| \right) \geq \\ &\geq \frac{2d-1}{4} \left(\pi |\vec{v}_1| + \frac{2d-1}{8} |\vec{v}_{23}| \right) \geq \frac{2d-1}{4} (\pi |\vec{v}_1| + \lambda |\vec{v}_{23}|) \geq \frac{2d-1}{4} |\vec{v}_{23}'|, \end{aligned}$$

since $\frac{2d-1}{8} \geq \frac{1}{4}$. Therefore, $\begin{pmatrix} \vec{v}_1' \\ \vec{v}_{23}' \end{pmatrix} \in C_{F(q)}^u$ and $DF(C_q^u) \subset C_{F(q)}^u$. As a consequence,

$$DF^k(C_{F^{-k}(q)}^u) \subset DF^{k-1}(C_{F^{-k+1}(q)}^u) \subset \cdots \subset DF(C_{F^{-1}(q)}^u) \subset C_q^u \quad \text{for any } k \in \mathbb{N}.$$

To prove that the intersection of this nested cones is a line, take

$$\begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ \vec{w}_{23} \end{pmatrix} \in C_{F^{-k}(q)}^u, \quad \begin{pmatrix} \vec{v}_1^k \\ \vec{v}_{23}^k \end{pmatrix} = DF^k \begin{pmatrix} \vec{v}_1 \\ \vec{v}_{23} \end{pmatrix}, \quad \begin{pmatrix} \vec{w}_1^k \\ \vec{w}_{23}^k \end{pmatrix} = DF^k \begin{pmatrix} \vec{w}_1 \\ \vec{w}_{23} \end{pmatrix}.$$

Put by definition, $|\vec{v}_1^j| = v_1^j$, $|\vec{w}_1^j| = w_1^j$, $\vec{v}_1 = (v_1, 0)$, $\vec{w}_1 = (w_1, 0)$, $v_1 > 0$, $w_1 > 0$. Then

$$\begin{aligned} \left| \frac{\vec{v}_{23}^1}{v_1^1} - \frac{\vec{w}_{23}^1}{w_1^1} \right| &= \left| \frac{\pi i \vec{v}_1 \exp 2\pi i t + \lambda \vec{v}_{23}}{(2d-1)v_1} - \frac{\pi i \vec{w}_1 \exp 2\pi i t + \lambda \vec{w}_{23}}{w_1} \right| = \\ &= \left| \frac{\pi i \exp 2\pi i t (w_1 \vec{v}_1 - v_1 \vec{w}_1)}{(2d-1)v_1 w_1} + \frac{\lambda}{2d-1} \left(\frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1} \right) \right| = \frac{\lambda}{2d-1} \left| \frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1} \right|, \end{aligned}$$

since $w_1 \vec{v}_1 - v_1 \vec{w}_1 = |\vec{w}_1| \vec{v}_1 - |\vec{v}_1| \vec{w}_1 = 0$. Therefore,

$$\left| \frac{\vec{v}_{23}^k}{v_1^k} - \frac{\vec{w}_{23}^k}{w_1^k} \right| = \left(\frac{\lambda}{2d-1} \right)^k \left| \frac{\vec{v}_{23}}{v_1} - \frac{\vec{w}_{23}}{w_1} \right|,$$

which goes to 0 as k goes to ∞ . Since the difference of slopes goes to 0, the cones converge to a line, say \mathbb{E}^u . The calculation gives that the restriction of the derivative DF on \mathbb{E}^u is an expansion. \square

Taking in mind the realization part of the proof of Theorem 2, we see that it is sufficient to construct the corresponding family of d -endomorphisms $S^1 \rightarrow S^1$, $d \geq 2$. First, we represent the two parameter family of circle endomorphisms $f_{\varepsilon, \delta}$ continuously depending on the parameters $\varepsilon \in (0; 1)$ and $\delta \in [0; \frac{1}{4}]$.

Let $U_\delta(x)$ be the bump-function such that

- $U_\delta(x) = 1$ for $x \in [-\frac{\delta}{2}; +\frac{\delta}{2}]$, $0 < \delta \leq \frac{1}{4}$;
- $U_\delta(x) = 0$ for $|x| \geq \delta$;
- $U'_\delta(x) \geq 0$ for $x \in [-\delta; -\frac{\delta}{2}]$, and $U'_\delta(x) \leq 0$ for $x \in [\frac{\delta}{2}; \delta]$.

Lemma 6 *Let*

$$f_{\varepsilon, \delta}(x) = \begin{cases} dx + (-d + \varepsilon)xU_\delta(x) \mod 1 & \text{for } \varepsilon \in (0; 1), \delta \in (0; \frac{1}{4}) \\ dx \mod 1 & \text{for } \varepsilon = 0, \delta = 0 \end{cases}$$

Then $f_{\varepsilon, \delta}$ is a structurally stable nonsingular circle d -endomorphism such that the non-wandering set $NW(f_{\varepsilon, \delta})$ is the union of a unique hyperbolic attracting point $x = 0$ and a Cantor set provided $\varepsilon \neq 0$ and $\delta \neq 0$. Moreover, $NW(f_{0,0}) = S^1$. In addition, $f_{\varepsilon, \delta} \rightarrow E_d$ as $\varepsilon \neq 0$ is fixed and $\delta \rightarrow 0$ in the C^0 topology.

Proof. For $\varepsilon \neq 0$ and $\delta \neq 0$, we see

$$f'_{\varepsilon, \delta}(x) = d + (-d + \varepsilon)[xU_\delta(x)' + U_\delta(x)] = d + (-d + \varepsilon)xU_\delta(x)' + (-d + \varepsilon)U_\delta(x).$$

Clearly, $d + (-d + \varepsilon)U_\delta(x) \geq \varepsilon$. Since $xU_\delta(x)' \leq 0$, $f'_{\varepsilon, \delta}(x) \geq \varepsilon$. Because of outside of the δ -neighborhood $V_\delta(0)$ of $x_0 = 0$ the mapping $f_{\varepsilon, \delta}$ coincides with the linear d -endomorphism $E_d(x) = dx \mod 1$, $f_{\varepsilon, \delta}$ is a nonsingular d -endomorphism. Since $f'_{\varepsilon, \delta}(0) = \varepsilon \in (0; 1)$, $x = 0$ is a hyperbolic attracting point. Solving the equation $dx + (-d + \varepsilon)xU_\delta(x) = x$, one gets two fixed points $\pm x_* \in V_\delta(0)$ such that $U_\delta(\pm x_*) = \frac{d-1}{d-\varepsilon}$, where $\frac{\delta}{2} < x_* < \delta$. Moreover, the ω -limit set of any point from $(-x_*, x_*)$ is $x_0 = 0$. Hence, $NW(f_{\varepsilon, \delta})$ equals

$$NW(f_{\varepsilon, \delta}) = \{x_0\} \bigcup (S^1 \setminus \cup_{k \geq 0} f_{\varepsilon, \delta}^{-k}(-x_*, x_*)),$$

where $C = S^1 \setminus \cup_{k \geq 0} f_{\varepsilon, \delta}^{-k}(-x_*, x_*)$ is Cantor set. For any $x \in C$, one can prove that

$$\begin{aligned} f'_{\varepsilon, \delta}(x) &= d + (-d + \varepsilon)xU'_\delta(x) + (-d + \varepsilon)U_\delta(x) \geq d + (-d + \varepsilon)U_\delta(x_*) + (-d + \varepsilon)xU'_\delta(x) = \\ &= 1 + (-d + \varepsilon)xU'_\delta(x) > 1. \end{aligned}$$

It follows from [15] that $f_{\varepsilon, \delta}$ is structurally stable. At last, for $x \in V_\delta(0)$, one gets

$$|f_{\varepsilon, \delta}(x) - E_d(x)| = |(-d + \varepsilon)xU_\delta(x)| \leq \delta d \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

As a consequence, $f_{\varepsilon, \delta} \rightarrow E_d$ as $\delta \rightarrow 0$ in the C^0 topology. \square

Taking in mind Lemma 6 and using the technics developed in [5] (see also [2, 3, 11, 23]), one can prove Theorem 3.

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